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Non-commutative analogues of q-special polynomials and a q-integral on a quantum sphere

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Abstract. q-Legendre polynomials can be treated as some special 'functions in the quantum double cosets $U(1)\backslash SU_q(2)/U(1)$'. They form a family (depending on a parameter q) of polynomials in one variable. We get their further generalization by introducing a two-parameter family of polynomials. If the former family arises from an algebra which is in a sense 'q-commutative', the latter one is related to its non-commutative counterpart. We also introduce a two-parameter deformation of the invariant integral on a quantum sphere.

1. Introduction

It is well known that the classical Legendre polynomials form a basis in the function space on the double cosets $U(1)\backslash SU(2)/U(1)$ [Vi]. Although q-analogues of these polynomials (as well as those of some other special functions) have been known for a long time, it became clear only recently that they can be treated as 'functions on quantum double cosets'.

This approach suggested in [VS] for the $sl(2)$ case (cf also [KN]) and developed by a number of authors for other quantum double cosets (cf., for example, survey [Va]) can be represented as follows. Let us consider the function space $\mathrm{Fun}_q(S^2)$ on the quantum sphere. This space can be defined as in the spirit of [P] as the subspace of left (or right) $U(1)$-invariant functions on $SU_q(2)$ (the group $U(1)$ can be treated as a commutative and cocommutative Hopf subalgebra of $SU_q(2)$).

For a generic q the space $\mathrm{Fun}_q(S^2)$ can be decomposed into a direct sum $\oplus \boldsymbol{V}_i$, $i = 0, 1, \ldots$ of irreducible $U_q(su(2))$-modules $\boldsymbol{V}_i$, where i is the spin (note that $\dim \boldsymbol{V}_i = 2i+1$). Let v be a generator of the subalgebra of $\mathrm{Fun}_q(S^2)$ formed by two-sided $U(1)$-invariant functions. Then the kth q-Legendre polynomial can be defined as a polynomial in v belonging to the component $\boldsymbol{V}_k$. In fact, the q-Legendre polynomials are nothing but eigenfunctions of the quantum Casimir operator.

It should be noted that the algebra $\mathrm{Fun}_q(S^2)$, which plays a crucial role in the constructions of [VS], is a particular case of a two-parameter family of $U_q(su(2))$-invariant (in the sense explained in section 2) associative algebras (meanwhile, the algebra $\mathrm{Fun}_q(S^2)$ itself depends only on the parameter q assuming that the parameter c labelling the orbits is fixed, see later).

This two-parameter family arises from a quantization of the Poisson pencil generated by the Kirillov–Kostant–Souriau (KKS) bracket on the usual sphere and the so-called R-matrix bracket (see section 6 for the definition).

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The quantization of the R-matrix bracket leads to the algebra $\mathrm{Fun}_q(S^2)$ which plays the role of 'commutative algebra' in the category of $U_q(sl(2))$-invariant algebras. The passage to the two-parameter family mentioned above is a way to a 'q-noncommutative' analysis. Our main aim is to apply the above approach to this family. More precisely, we introduce $(\hbar, q)$-special polynomials as eigenfunctions of the quantum Casimir operator acting on this two-parameter family.

Moreover, we introduce as in [NM], a certain $(\hbar, q)$-analogue of the invariant integral on the sphere (the authors of [NM] deal with a q-analogue of the integral which is defined on the algebra $\mathrm{Fun}_q(S^2)$). Finally, we give an explicit expression for this $(\hbar, q)$-integral which is a generalization of the well known Jackson integral (cf, for example, [VS], [KN]).

Throughout this paper, the basic field is $\mathbb{C}$ and $q \in \mathbb{C}$ is assumed to be generic. Thus, we deal with the group $SL(2, \mathbb{C})$, the complexification of the sphere and their quantum counterparts rather then with compact objects themselves: the reason for this is explained in section 4.

The paper is organized as follows. In section 2 we define our basic object: a function algebra on a quantum hyperboloid. In section 3 we compute the action of the quantum Casimir operator on the elements of this algebra.

In section 4 we define $(\hbar, q)$-special polynomials in the above algebra.

Section 5 is devoted to introducing an $(\hbar, q)$-analogue of the invariant integral on a sphere. In section 6 we discuss the considered objects in the framework of deformation quantization. There we also explain in what sense we use the term *a q-commutative algebra*.

2. Basic objects: the algebra $A^c_{\hbar,q}$

Let us consider the quantum enveloping algebra $U_q(sl(2))$, i.e. an algebra generated by elements E_+, E_-, X, Y satisfying the following relations:

$$E_\pm X = q^{\pm 1} X E_\pm \qquad E_\pm Y = q^{\mp 1} Y E_\pm \qquad E_+ E_- = E_- E_+ = 1$$
$$[X, Y] = \frac{E_+^2 - E_-^2}{q - q^{-1}}$$

where $q \neq 0$, $q^2 \neq 1$, equipped with the coproduct

$$\Delta(X) = E_- \otimes X + X \otimes E_+ \qquad \Delta(Y) = E_- \otimes Y + Y \otimes E_+ \qquad \Delta(E_\pm) = E_\pm \otimes E_\pm$$

and some antipode whose explicit form we do not need.

One can verify that the element, called the *quantum Casimir operator* or simply quantum Casimir,

$$K = \frac{q}{2}(XY + YX) + \frac{q^2(1+q^2)}{2(1-q^2)^2}(E_+^2 + E_-^2 - 2)$$

belongs to the centre of the above algebra.

It is well known that the image of the quantum Casimir in any irredicible $U_q(sl(2))$-module is a scalar operator. Let us denote by λ_k, $k = 0, 1/2, 1, \ldots$, its eigenvalue corresponding to an irreducible spin k $U_q(sl(2))$-module $\boldsymbol{V}_k$. We will show later that

$$\lambda_k = \frac{(q^{2k} - 1)(q^{2(k+1)} - 1)}{q^{2k-2}(q^2 - 1)^2}. \tag{1}$$

Now, let us consider a three-dimensional $U_q(sl(2))$-module $\boldsymbol{V} = \boldsymbol{V}_1$ such that the representation $\rho_q : U_q(sl(2)) \to \mathrm{End}(\boldsymbol{V})$ coincides with the classical one $\rho\colon U(sl(2)) \to$

End($\boldsymbol{V}$) as $q = 1$. Let us fix a basis $\{u, v, w\}$ in $\boldsymbol{V}$ such that the above action of the quantum group is given by (we omit the symbol ρ_q in our notation):

$$E_\pm u = q^{\pm 1}u \qquad E_\pm v = v \qquad E_\pm w = q^{\mp 1}w \qquad Xu = 0 \qquad Xv = -(q+q^{-1})u$$
$$Xw = v \qquad Yu = -v \qquad Yv = (q+q^{-1})w \qquad Yw = 0.$$

By the coproduct, we can equip $\boldsymbol{V}^{\otimes 2}$ with a $U_q(sl(2))$-module structure as well. This module is reducible and can be decomposed into a direct sum of three irreducible $U_q(sl(2))$-modules

$$\boldsymbol{V}_0 = \text{span}\left\{(q^2+1)uw + vv + \frac{q^2+1}{q^2}wu\right\}$$
$$\boldsymbol{V}_1 = \text{span}\{q^2uv - vu, (q^2+1)(uw - wu) + (1-q^2)vv, -q^2vw + wv\}$$
$$\boldsymbol{V}_2 = \text{span}\{uu, uv + q^2vu, q^{-1}uw - qvv + q^3wu, vw + q^2wv, ww\}$$

of spins 0,1,2, respectively (hereafter the sign $\otimes$ is omitted).

Then only the following relations imposed on the elements of the space $\boldsymbol{V}^{\otimes 2} \oplus \boldsymbol{V} \oplus \mathbb{C}$ are consistent with the above action of $U_q(sl(2))$:

$$C_q = (q^2+1)uw + vv + \frac{q^2+1}{q^2}wu = c$$
$$q^2uv - vu = -\overline{h}u$$
$$(q^2+1)(uw - wu) + (1-q^2)vv = \overline{h}v$$
$$-q^2vw + wv = \overline{h}w$$

with arbitrary $\overline{h}$ and c. The element C_q is called *a braided Casimir*.

Let us denote $A^c_{h,q}$ the quotient algebra of a free tensor algebra $T(\boldsymbol{V})$ by the ideal generated by the elements

$$(q^2+1)uw + vv + \frac{q^2+1}{q^2}wu - c \qquad q^2uv - vu + \overline{h}u$$
$$(q^2+1)(uw - wu) + (1-q^2)vv - \overline{h}v \qquad -q^2vw + wv - \overline{h}w.$$

Remark 1. Here $\overline{h}$ and q are assumed to be fixed. If we want to consider them as formal parameters, we must replace $T(\boldsymbol{V})$ in the definition of the algebra $A^c_{h,q}$ by $T(\boldsymbol{V}) \otimes \mathbb{C}[[\overline{h}, q, q^{-1}]]$. The parameter c which labels the orbits is always fixed. The case $c = 0$ corresponds to the cone.

If $q = 1$, $\overline{h} = 0$, we get a family (parametrized by the parameter c which labels the orbits) of usual hyperboloids considered as orbits in $sl(2)^*$. If $q = 1$, $\overline{h} \neq 0$, we get its non-commutative analogue but it still lives in the classical category of $sl(2)$-invariant algebras.

If $q \neq 1$, we get a two-parameter family of $U_q(sl(2))$-invariant algebras. Let us recall that an associative algebra A is called $U_q(g)$-invariant (or covariant) if

$$X \circ (a \otimes b) = \circ \Delta\, X(a \otimes b) \qquad \forall X \in U_q(g), a, b \in A$$

where $\circ$ is the product in A.

In fact, the Podles' quantum spheres are exactly these quantum hyperboloids equipped with an involution. Here, we would like to avoid a discussion of the problem of a proper definition of an involution in braided categories (it has been discussed in [DGR1]) and prefer to work with complex objects.

The particular case $\overline{h} = 0$ of this family corresponds to a q-commutative algebra in the sense discussed in section 6.

Let us rewrite the above equations as follows:

$$(q^2+1)uw + \tilde{v}^2 + \frac{q^2+1}{q^2}wu = \tilde{c} - 2a\tilde{v}$$
$$q^2u\tilde{v} - \tilde{v}u = 0$$
$$(q^2+1)(uw - wu) + (1-q^2)\tilde{v}^2 = -\overline{h}\tilde{v}$$
$$-q^2\tilde{v}w + w\tilde{v} = 0$$

where $a = \overline{h}(1-q^2)^{-1}$, $\tilde{c} = c - a^2$, $\tilde{v} = v - a$.

By these relations we can express the product uw in terms of the variable $\tilde{v}$:

$$uw = (q^2+1)^{-2}[\tilde{c}q^2 - a(1+q^2)\tilde{v} - \tilde{v}^2]. \tag{2}$$

We will apply this formula later.

3. Action of the quantum Casimir

Our next aim is to obtain a formula for $K\tilde{v}^k$ for every natural k, where $\tilde{v}^k = \tilde{v}^{\otimes k}$.

It is clear from the very beginning that the action of $E_+^2 + E_-^2 - 2$ on $\tilde{v}^k$ equals zero. On the other hand, from the relation of commutation for X, Y it is also clear that the actions of XY and YX on $\tilde{v}^k$ coincide.

Thus, we have

$$K\tilde{v}^k = qYX\tilde{v}^k.$$

By the formulae for the coproduct and for the action of X, E_+, E_- on $\tilde{v}$, as well as the formula of commutation of v and $\tilde{v}$, we have

$$\begin{aligned} X\tilde{v}^k &= (XE_+^{k-1} + E_-XE_+^{k-2} + \cdots + E_-^{k-1}X)\tilde{v}^k \\ &= -\frac{q^2+1}{q}(u\tilde{v}^{k-1} + \tilde{v}u\tilde{v}^{k-2} + \cdots + \tilde{v}^{k-1}u) \\ &= -\frac{q^2+1}{q}(1+q^2+\cdots+q^{2(k-1)})u\tilde{v}^{k-1} = -\alpha_k(q)u\tilde{v}^{k-1} \end{aligned}$$

with

$$\alpha_k(q) = \frac{(q^2+1)(q^{2k}-1)}{q(q^2-1)}.$$

Hereafter by $XE_+^{k-1}\tilde{v}^k$ we mean $X\tilde{v}(E_+\tilde{v})^{k-1}$ etc.

Similarly, in virtue of the formula (2) we get

$$\begin{aligned} YX\tilde{v}^k &= -\alpha_k(q)(YE_+^{k-1} + E_-YE_+^{k-2} + \cdots + E_-^{k-1}Y)u\tilde{v}^{k-1} \\ &= -\alpha_k(q)\left[-(\tilde{v}+a)\tilde{v}^{k-1} + \frac{q^2+1}{q^2}(uw\tilde{v}^{k-2} + u\tilde{v}w\tilde{v}^{k-3} + \cdots + u\tilde{v}^{k-2}w)\right] \\ &= \alpha_k(q)\left[\tilde{v}^k + a\tilde{v}^{k-1} - \frac{q^2+1}{q^2}(1+q^{-2}+\cdots+q^{-2(k-2)})uw\tilde{v}^{k-2}\right] \\ &= \alpha_k(q)\left[\tilde{v}^k + a\tilde{v}^{k-1} + \frac{q^{-2(k-1)}-1}{q^2(q^2+1)(q^{-2}-1)}(\tilde{v}^2 + a(q^2+1)\tilde{v} - \tilde{c}q^2)\tilde{v}^{k-2}\right] \end{aligned}$$

$$= \alpha_k(q)\left[\tilde{v}^k + a\tilde{v}^{k-1} + \frac{q^{2(k-1)}-1}{q^{2(k-1)}(q^2+1)(q^2-1)}(\tilde{v}^k + a(q^2+1)\tilde{v}^{k-1} - \tilde{c}q^2\tilde{v}^{k-2})\right]$$
$$= \beta_k(q)\left[\frac{q^{2(k+1)}-1}{q^2-1}\tilde{v}^k + a(q^2+1)\frac{q^{2k}-1}{q^2-1}\tilde{v}^{k-1} - \tilde{c}q^2\frac{q^{2(k-1)}-1}{q^2-1}\tilde{v}^{k-2}\right]$$

with

$$\beta_k(q) = \frac{\alpha_k(q)}{q^{2(k-1)}(q^2+1)} = \frac{q^{2k}-1}{q^{2k-1}(q^2-1)}$$

(we assume that $\tilde{v}^{-1} = \tilde{v}^{-2} = 0$).

Thus, we have established the following.

Proposition 1.

$$K\tilde{v}^k = q\beta_k(q)\left[\frac{q^{2(k+1)}-1}{q^2-1}\tilde{v}^k + a(q^2+1)\frac{q^{2k}-1}{q^2-1}\tilde{v}^{k-1} - \tilde{c}q^2\frac{q^{2(k-1)}-1}{q^2-1}\tilde{v}^{k-2}\right].$$

Remark 2. This proposition generalizes proposition 6.2 from [VS]. However, in order to represent it in a form similar to that from [VS], let us introduce the notion of right and left *q-difference* for a function $f(z)$ ($z \in \mathbb{C}$) as follows:

$$\delta^+{}_q f(z) := \frac{f(z) - f(qz)}{z - qz} \qquad \delta^-{}_q f(z) := \frac{f(z) - f(q^{-1}z)}{z - q^{-1}z}.$$

In particular, for $f(z) = z^k$ we have

$$\delta^+{}_{q^2} z^k = z^{k-1}\frac{q^{2k}-1}{q^2-1} \qquad \delta^-{}_{q^2} z^k = z^{k-1}\frac{q^{2k}-1}{q^{2(k-1)}(q^2-1)}.$$

By this notation, we can rewrite the above formula for the action of the Casimir as follows:

$$K\tilde{v}^k = \frac{q^{2k}-1}{q^{2(k-1)}(q^2-1)}\delta^+{}_{q^2}[(\tilde{v}^2 + a(q^2+1)\tilde{v} - \tilde{c}q^2)\tilde{v}^{k-1}]$$
$$= [\delta^+{}_{q^2}(\tilde{v}^2 + a(q^2+1)\tilde{v} - \tilde{c}q^2)\delta^-{}_{q^2}]\tilde{v}^k$$
$$= \delta^+{}_{q^2}\left(\tilde{v}^2 + h\frac{1+q^2}{1-q^2}\tilde{v} - \tilde{c}q^2\right)\delta^-{}_{q^2}\tilde{v}^k.$$

Thus, the action of the Casimir operator on any polynomial in $\tilde{v}$ can be expressed in terms of the q^2-difference operator of second order.

4. $(\bar{h}, q)$-special polynomials

It is well known that the function algebra $A^c_{0,1}$ on a usual hyperboloid considered as an algebraic variety in $sl(2)^*$ is a direct sum of all integer spin irreducible $sl(2)$-modules $\boldsymbol{V}_k$. This property is also valid for its non-commutative analogue $A^c_{h,1}$. It is also true if q is generic for a $U_q(sl(2))$-invariant algebra $A^c_{h,q}$.

To show this it suffices to check that for any integer spin k there exists in the algebra $A^c_{h,q}$ a unique polynomial in $\tilde{v}$ belonging to the module $\boldsymbol{V}_k$ (cf [DG], where another method of proof is given).

This property is ensured by the following.

Proposition 2. For any λ_k, $k = 0, 1, 2, \ldots$, given by the formulae (1) and for a generic q there exists a unique polynomial of the form

$$P_k(\tilde{v}) = \sum_{j=0}^{k} A_j^k \tilde{v}^{k-j} \qquad \text{with } A_0^k = 1$$

such that

$$KP_k(\tilde{v}) = \lambda_k P_k(\tilde{v}).$$

Proof. Let $P_k(\tilde{v})$ be such a polynomial.

By proposition 1 we have

$$\begin{aligned} KP_k(\tilde{v}) &= \sum_{j=0}^{k} A_j^k (a_{k-j}\tilde{v}^{k-j} + b_{k-j}\tilde{v}^{k-j-1} + c_{k-j}\tilde{v}^{k-j-2}) \\ &= a_k\tilde{v}^k + (b_k + A_1^k a_{k-1})\tilde{v}^{k-1} \\ &\quad + \sum_{j=2}^{k} (A_{j-2}^k c_{k-j+2} + A_{j-1}^k b_{k-j+1} + A_j^k a_{k-j})\tilde{v}^{k-j} \end{aligned}$$

where

$$a_k = \frac{(q^{2k}-1)(q^{2(k+1)}-1)}{q^{2k-2}(q^2-1)^2} \qquad b_k = \frac{a(q^2+1)(q^{2k}-1)^2}{q^{2k-2}(q^2-1)^2}$$

$$c_k = -\frac{\tilde{c}(q^{2k}-1)(q^{2(k-1)}-1)}{q^{2k-4}(q^2-1)^2}.$$

Now the above equality of polynomials gives us the following chain of relations:

$$a_k = \lambda_k \qquad b_k + A_1^k a_{k-1} = A_1^k \lambda_k$$

$$A_{j-2}^k c_{k-j+2} + A_{j-1}^k b_{k-j+1} + A_j^k a_{k-j} = A_j^k \lambda_k \qquad (j = 2, 3, \ldots, k).$$

So, we have the following recurrence relations for finding A_j^k:

$$A_1^k = \frac{b_k}{a_k - a_{k-1}}$$

$$A_j^k = \frac{A_{j-2}^k c_{k-j+2} + A_{j-1}^k b_{k-j+1}}{a_k - a_{k-j}} \qquad (j = 2, 3, \ldots, k).$$

It remains to say that the numerators of these formulae are not equal to zero for a generic q. This completes the proof. □

Let us remark that this approach gives us a description of 'non-generic' values of q: they are exactly such that the numerators of the above formulae vanish. It should be noted that these numerators do not contain $\overline{h}$ and therefore the decomposition $A_{h,q}^c = \oplus \boldsymbol{V}_k$ is valid for a generic q independently on $\overline{h}$.

We call the above polynomials $(\overline{h}, q)$-*special polynomials*. If $\overline{h} = 0$ and $q = 1$, they coincide with the Legendre polynomials up to a change of the variable and up to factors. A change of the variable consisting in multiplying the variable by $\sqrt{-1}$ is motivated by the fact that the Legendre polynomials arise from the real compact form of the group $SL(2, \mathbb{C})$.

Since the Legendre polynomials are even for even k and odd for odd k, this substitution leads to polynomials with real coefficients (for an odd k it is necessary also to multiply the polynomial by $\sqrt{-1}$).

It is still true if $\hbar = 0$ but $q \neq 1$. Thus, assuming q to be real, in a similar way we get the polynomials with real coefficients which differ from the q-Legendre polynomials by factors (cf [VS], [KN], [Va]) .

However, if $\hbar \neq 0$ and $q \neq 1$, the above property is no longer true and the mentioned procedure does not lead to polynomials with real coefficients.

This is the reason why we do prefer to deal with the complex form of the quantum hyperboloid.

5. $(\hbar, q)$-integral

Let us introduce in the algebra $A^c_{\hbar,q}$ an analogue of the invariant integral. It is exactly the projector in this algebra onto its trivial component.

In what follows we use the notation Int: $A^c_{\hbar,q} \to \mathbb{C}$ for it. If $q = 1, \hbar = 0$, this operator coincides up to a factor with the usual invariant integral on a sphere.

In the general case, we call this projector an *$(\hbar, q)$-integral*.

Our immediate aim is to compute the values $\mathrm{Int}(\tilde{v}^k)$. We use the method of [NM], where a particular case ($\hbar = 0$) has been considered.

It is obvious that $\mathrm{Int}(Yf) = 0$ for any $f \in A^c_{\hbar,q}$. This follows from the fact that $Yf \in \boldsymbol{V}_k$ if $f \in \boldsymbol{V}_k, k \neq 0$ and $Yf = 0$ if $f \in \boldsymbol{V}_0$.

Let us set $f = u\tilde{v}^k$. Then, applying again the formula (2), we have

$$\begin{aligned}\mathrm{Int}(Yu\tilde{v}^k) = (YE^k_+ + E_-YE^{k-1}_+ + \cdots + E^k_-Y)u\tilde{v}^k \\ &= -(\tilde{v}+a)\tilde{v}^k + q^{-1}(q+q^{-1})u(w\tilde{v}^{k-1} + \tilde{v}w\tilde{v}^{k-2} + \cdots + \tilde{v}^{k-1}w) \\ &= -\tilde{v}^{k+1} - a\tilde{v}^k + (1+q^{-2})uw(1+q^{-2}+\cdots+q^{-2(k-1)})\tilde{v}^{k-1} \\ &= -\tilde{v}^{k+1} - a\tilde{v}^k + (q^2-1)^{-1}(1-q^{-2k})(1+q^2)^{-1} \\ &\quad \times(-\tilde{v}^2 - a(q^2+1)\tilde{v} + \tilde{c}q^2)\tilde{v}^{k-1} = 0.\end{aligned}$$

This implies the following equation

$$\mu_{k+1}(q^{2k+4}-1) + \mu_k a(q^{2k+2}-1)(1+q^2) - \mu_{k-1}(q^{2k}-1)q^2\tilde{c} = 0$$

where $\mu_k = \mathrm{Int}(\tilde{v}^k)$. Now by putting $\gamma_k = \mu_k(q^{2k+2}-1)$ we have

$$\gamma_{k+1} + a(1+q^2)\gamma_k - q^2\tilde{c}\gamma_{k-1} = 0.$$

Thus, if we normalize the $(\hbar, q)$-integral by $\mathrm{Int}(1) = 1$ and $\mathrm{Int}(v) = 0$, we have for μ_k the following formula:

$$\mu_k = (q^2-1)(q^{2k+2}-1)^{-1}(y_2x_1^k - y_1x_2^k)(x_2-x_1)^{-1} \tag{3}$$

where $y_i = x_i + a(q^2+1)$ and x_1 and x_2 are the roots of the quadratic equation

$$x^2 + a(1+q^2)x - q^2\tilde{c}^2 = 0.$$

Thus, we have proved the following.

Proposition 3. The $(\hbar, q)$-integral normalized by $\mathrm{Int}(1) = 1$ and $\mathrm{Int}(v) = 0$ is unique and defined by the formula $\mathrm{Int}(\tilde{v}^k) = \mu_k$, where μ_k is given by (3).

Assuming $|q|$ to be smaller than 1, we can represent this formula as

$$\mathrm{Int}(f) = (1-q^2)(x_2-x_1)^{-1}\sum_{m=0}^{\infty}(y_2f(x_1q^{2m}) - y_1f(x_2q^{2m}))q^{2m}$$

where f is a polynomial in $\tilde{v}$.

Remark 3. (a) Let us remark that for the polynomials $P_k(\tilde{v}))$, $k \geqslant 0$ introduced in section 4 we have

$$\mathrm{Int}(P_k(\tilde{v})P_l(\tilde{v})) = 0 \qquad (k \neq l)$$

i.e. $(\overline{h}, q)$-special polynomials are mutually orthogonal with respect to the pairing defined by the $(\overline{h}, q)$-integral. This follows from the fact that in the decomposition $\boldsymbol{V}_k \otimes \boldsymbol{V}_l$ into a direct sum of irreducible components, the trivial component is present if and only if $k = l$.

(b) Let us note that neither our formula for $(\overline{h}, q)$-special polynomials nor that for the $(\overline{h}, q)$-integral have any limit as $q \to 1$ if $\overline{h} \neq 0$. In the classical case $(q = 1)$ one usually deals with a family of finite-dimensional representations of the algebras $A^c_{h,1}$. In such a representation, 'the $(\overline{h}, 1)$-integral' becomes a usual trace (up to a factor).

(c) If $\overline{h} = 0, q \neq 0$, the above formula for the $(\overline{h}, q)$-integral gives the well known formula for the Jackson integral (cf [VS], [KN]). The relations of orthogonality for $(\overline{h}, q)$-special polynomials with respect to the $(\overline{h}, q)$-integral generalize those for q-Legendre polynomials with respect to the Jackson integral.

6. Connection with the deformation quantization

In fact, we have shown that the deformation $A^c_{0,1} \to A^c_{h,q}$ is flat and therefore one can introduce the corresponding quasiclassical object. This is *a Poisson pencil* (i.e. a linear space of Poisson brackets) generated by the KKS bracket and a so-called *R-matrix bracket* well defined on a hyperboloid.

The latter bracket is introduced by $\{f, g\} = \mu\langle\rho^{\otimes 2}(R), \mathrm{d}f \otimes \mathrm{d}g\rangle$, where R is the unique (up to a factor and an intertwinning) solution of the classical *modified* Yang–Baxter equation on the Lie algebra $sl(2)$, ρ is the coadjoint representation restricted to a hyperboloid, and we use the pairing between vector fields and differential forms (extended onto their tensor powers)†. μ is the commutative product in the function space $A^c_{0,1}$.

Thus, the algebra $A^c_{h,q}$ can be treated as a quantum object with respect to the above Poisson pencil. Let us emphasize that the quantization of the only KKS bracket gives the algebra $A^c_{h,1}$ which is still $sl(2)$-invariant. Let us introduce an $sl(2)$-morphism $\phi : A^c_{0,1} \to A^c_{h,1}$ by sending $u^k \in A^c_{0,1}$ to $u^k \in A^c_{h,1}$.

By means of this morphism we can, in the spirit of the deformation quantization theory, introduce a new $sl(2)$-invariant associative product in the algebra $A^c_{0,1}$:

$$a \circ_{\overline{h}} b = \phi^{-1}(\phi(a) \circ \phi(b)) \qquad a, b \in A^c_{0,1}$$

where $\circ$ is the product in the algebra $A^c_{h,1}$. One can see that this quantization is closed in the sense of [CFS] (this means that the trace in the quantum algebra is exactly an integral on the initial manifold, in our case such a manifold is a sphere).

Let us remark that a deformation quantization exists for any symplectic Poisson bracket on any (compact smooth) manifold.

The passage $A^c_{0,1} \to A^c_{h,1}$ is a particular case of this deformation quantization scheme since the KKS bracket is symplectic. It is not the case of the R-matrix bracket: it is not symplectic and its quantization leads to a deformation of the integral. Although it is easy, by a method similar to the above, to represent the algebra $A^c_{h,q}$ as $A^c_{0,1}$ equipped with a deformed product $\circ_{\overline{h},q}$, the initial integral on $A^c_{0,1}$ is not any more a trace for this product.

† As for other simple Lie algebras g such a type of Poisson pencil exists only on some exceptional orbits in g^* (cf [GP]).

Thus, the first step of the quantization, i.e. the passage $A^c_{0,1} \rightarrow A^c_{h,1}$, can be done without any deformation of the integral. In contrast, the second one, i.e. the further passage to the algebra $A^c_{h,q}$, leads to such a deformation.

Let us explain now in what sense we use the term *q-commutative* for the algebra $A^c_{0,q}$. In this algebra there exists an involutive ($\tilde{S}^2 = \mathrm{id}$) operator $\tilde{S}: (A^c_{0,q})^{\otimes 2} \rightarrow (A^c_{0,q})^{\otimes 2}$ which plays the role of the ordinary flip in the algebra $A^c_{0,q}$. It can be derived from the Yang–Baxter operator S: it suffices to replace all eigenvalues of S close to 1 (respectively, -1) by 1 (respectively, -1) keeping all eigenspaces of S (assuming that $|q - 1| \ll 1$).

Another description of the operator $\tilde{S}$ is given in [DS]. Using the results of this paper, one can see that in the algebra $A^c_{0,q}$ we have $a \circ b = \circ(\tilde{S}(a \otimes b))$ for any two elements $a, b \in A^c_{0,q}$. In this sense, we say that the algebra $A^c_{0,q}$ is *q-commutative*.

Thus, quantizing the only R-matrix bracket, we pass from a commutative algebra to a q-commutative one. Meanwhile, a simultaneous quantization of the considered Poisson pencil leads to the algebras which are $U_q(g)$-invariant but are no longer q-commutative. This gives a simultaneous deformation of the category (instead of $sl(2)$-invariant algebras we get $U_q(g)$-invariant ones) and a passage from 'commutative' objects to 'non-commutative' ones in the new category.

We consider the final algebra $A^c_{h,q}$ as an object of twisted quantum mechanics, which looks like similar objects of super-quantum mechanics. For a more detailed discussion of this point of view, we refer the reader to [DGR1] and [DGR2].

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